

is important an analysis of second derivatives.

Second Derivative Test: Suppose f is
point of f .

differentiable at \bar{p} and \bar{p} is a critical

1.) If $f_{xx}(\bar{p}) > 0$ and $D(\bar{p}) = f_{xx}(\bar{p})f_{yy}(\bar{p}) - (f_{xy}(\bar{p}))^2 > 0$,
then f has a local min at \bar{p} .

2.) If $f_{xx}(\bar{p}) < 0$ and $D(\bar{p}) = f_{xx}(\bar{p})f_{yy}(\bar{p}) - (f_{xy}(\bar{p}))^2 > 0$ then f has
a local max at \bar{p} .

3.) If $D(\bar{p}) = f_{xx}(\bar{p})f_{yy}(\bar{p}) - (f_{xy}(\bar{p}))^2 < 0$, then f has a
saddle point at \bar{p} .

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LAST TIME 2nd Derivative Test

$$D = f_{xx}f_{yy} - (f_{xy})^2, \bar{p} \text{ a c.p.}$$

1. $f_{xx}(\bar{p}) > 0$ and $D(\bar{p}) > 0 \Rightarrow \bar{p}$ a local min point

2. $f_{xx}(\bar{p}) < 0$ and $D(\bar{p}) > 0 \Rightarrow \bar{p}$ a local max point

3. $D(\bar{p}) < 0 \Rightarrow \bar{p}$ is a saddle point.

Note.

① If $D(\bar{p}) = 0$, we cannot say anything about \bar{p} with this test.

② You could use $f_{yy}(\bar{p})$ in place of $f_{xx}(\bar{p})$

Ex. Classify the critical points of $f(x, y) = xy + e^{-xy}$ with the 2nd derivative test.

~~Ans~~ Sol. $\nabla f = \langle y - ye^{-xy}, x - xe^{-xy} \rangle$
 $= \langle y(1 - e^{-xy}), x(1 - e^{-xy}) \rangle$

$$\therefore \nabla f = \vec{0} \text{ iff } \begin{cases} y(1 - e^{-xy}) = 0 \\ x(1 - e^{-xy}) = 0 \end{cases}$$
$$\text{iff } \begin{cases} y = 0 \text{ or } 1 - e^{-xy} = 0 \\ x = 0 \text{ or } 1 - e^{-xy} = 0 \end{cases}$$

Note $e^{-xy} = 1$ iff $-xy = 0$ iff $x = 0$ or $y = 0$

$\therefore \nabla f = \vec{0}$ iff either $x = 0$ or $y = 0$

↳ because $x = 0$ or $y = 0$ implies both of the conditions for $\nabla f = \vec{0}$

~~Picture of CP~~

Now we need $D(x, y)$:

$$f_{xx} = \cancel{xy^2} \text{ and } y^2 e^{-xy} \quad f_{yy} = x^2 e^{-xy}$$

$$f_{xy} = f_{yx} = 1 - (e^{-xy} - xy e^{-xy}) = 1 - e^{-xy}(1 - xy)$$

$$\therefore D(x, y) = f_{xx} f_{yy} - (f_{xy})^2$$

$$= (y^2 e^{-xy})(x^2 e^{-xy}) - (1 - e^{-xy}(1 - xy))^2$$

$D(x, y) = 0$ uniformly when $x = 0$ or $y = 0$. (Verify)

\therefore Nothing can be said with 2nd derivative test.

Ex. Classify CPs of $f(x,y) = x^2 + y^2 + xy + y$ with 2nd derivative test.

Sol: $\nabla f = \langle 2x + y, 2y + x + 1 \rangle$

$\therefore \nabla f = \vec{0}$ iff $\begin{cases} 2x + y = 0 \\ 2y + x + 1 = 0 \end{cases}$ iff $\begin{cases} y = -2x \\ 2(-2x) + x + 1 = 0 \end{cases}$

iff $\begin{cases} -3x + 1 = 0 \\ y = -2x \end{cases}$ iff $\begin{cases} x = \frac{1}{3} \\ y = -\frac{2}{3} \end{cases}$

\therefore We have a unique CP at $(\frac{1}{3}, -\frac{2}{3})$.

$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2$ Here $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} + f_{yx} = 1$

$\therefore D(x,y) = 2 \cdot 2 - 1^2 = 3 > 0$

\therefore Because $f_{xx}(\frac{1}{3}, -\frac{2}{3}) = 2 > 0$ and $D(\frac{1}{3}, -\frac{2}{3}) = 3 > 0$ we have by 2nd derivative test $(\frac{1}{3}, -\frac{2}{3})$ is a local min point.

With local min point value $f(\frac{1}{3}, -\frac{2}{3}) = (\frac{1}{3})^2 + (-\frac{2}{3})^2 + (\frac{1}{3})(-\frac{2}{3}) + (-\frac{2}{3})$
 $= \frac{1}{9} + \frac{4}{9} - \frac{2}{9} - \frac{6}{9} = -\frac{1}{3}$

Ex: Classify CPs of $f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 4x$.

Sol: CPs are given when $\nabla f = \vec{0}$:

$\nabla f = \langle 3x^2 - 6x - 4, 3y^2 - 6y \rangle$
 $= 3 \langle x^2 - 2x - \frac{4}{3}, y^2 - 2y \rangle$
 $= 3 \langle (x-3)(x+1), y(y-2) \rangle$

	$x=3$	$x=-1$
$y=0$	$(3, 0)$	$(-1, 0)$
$y=2$	$(3, 2)$	$(-1, 2)$

$\therefore \nabla f = \vec{0}$ iff $\begin{cases} (x-3)(x+1) = 0 \\ y(y-2) = 0 \end{cases}$ iff $\begin{cases} x=3 \text{ or } x=-1 \\ y=0 \text{ or } y=2 \end{cases}$

\therefore We have CPs: $(3, 0), (-1, 0), (3, 2), (-1, 2)$.

Next we compute $D(x,y)$:

$f_{xx} = 6x - 6$, $f_{yy} = 6y - 6$, $f_{xy} = f_{yx} = 0$

$\therefore D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = 6(x-1) \cdot 6(y-1) - 0^2 = 36(x-1)(y-1)$

Now analyze P_s :

For $(3,0)$: $f_{xx}(3,0) = 6(3-1) > 0$ and $D(3,0) = 36(3-1)(0-1) < 0$

Because $D(3,0) < 0$, $(3,0)$ is a saddle point of f .

For $(-1,0)$: $f_{xx}(-1,0) = 6(-1-1) < 0$ and $D(-1,0) = 36(-1-1)(0-1) > 0$

$(-1,0)$ is a local max point of f with local max value $f(-1,0) = 5$

For $(3,2)$: $f_{xx}(3,2) = 6(3-1) > 0$ and $D(3,2) = 36(3-1)(2-1) > 0$

$(3,2)$ is a local min point of f with local min value $f(3,2) = -3$

For $(-1,2)$: $f_{xx}(-1,2) = 6(-1-1) < 0$ and $D(-1,2) = 36(-1-1)(2-1) < 0$

Because $D(-1,2) < 0$, $(-1,2)$ is a saddle point of f .

Section 7: Lagrange Multipliers

IDEA: Method for solving constrained optimization problems.

Observation: If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ has a level curve

$f(\bar{x}) = c$, then for every point \bar{p} on this level curve: $\nabla f(\bar{p}) = \nabla c = 0$
i.e. ∇f is orthogonal to level curves.

IDEA: Set up constrained optimization to be optimizations on a level curve.

In general:
$$\begin{cases} \text{Optimize } f(\bar{x}) \\ \text{Subject to } g_1(\bar{x}) = 0, g_2(\bar{x}) = 0, \dots \end{cases}$$

Can be turned into optimize: $f(\bar{x}) - \lambda_1 g_1(\bar{x}) - \lambda_2 g_2(\bar{x}) - \dots - \lambda_n g_n(\bar{x})$ ← 0-level curves

$$F(\bar{x}, \lambda_1, \lambda_2, \dots, \lambda_n) = f(\bar{x}) - \lambda_1 g_1(\bar{x}) - \lambda_2 g_2(\bar{x}) - \dots - \lambda_n g_n(\bar{x})$$

So $F(\bar{x}, \lambda)$ has CPs with $\nabla F = 0$

So Supposing the solution set of $g_1(\bar{x}) = 0 = g_2(\bar{x}) = \dots = g_n(\bar{x})$ is closed and bounded, the local extreme of F determine global extreme of f under $g_1 = g_2 = \dots = g_n = 0$

Now let's see this in Practice...

Ex: Optimize $f(x, y) = xe^y$ subject to $x^2 + y^2 = 2$

Sol: Via Lagrange Multipliers:

$$F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

with $g(x, y) = x^2 + y^2 - 2$ (because $g(x, y) = 0$ iff $x^2 + y^2 = 2$)

$$\text{So } F(x, y, \lambda) = xe^y - \lambda(x^2 + y^2 - 2)$$

$$\therefore \nabla F = \langle e^y - \lambda(2x), xe^y - \lambda(2y), -(x^2 + y^2 - 2) \rangle$$

$$\text{So } \nabla F = 0 \text{ iff } \begin{cases} e^y - \lambda(2x) = 0 \\ xe^y - \lambda(2y) = 0 \\ -(x^2 + y^2 - 2) = 0 \end{cases} \text{ iff } \begin{cases} 2\lambda x = e^y & (1) \\ 2\lambda y = xe^y & (2) \\ x^2 + y^2 = 2 & (3) \end{cases}$$

(Idea: Solve this system)

Now $2\lambda x = e^y$ implies $\lambda \neq 0$ (lest $e^y = 0$ which is nonsense)

~~$\therefore x = \frac{e^y}{2\lambda}$, so using equation 2: $2\lambda y = \left(\frac{e^y}{2\lambda}\right)e^y$~~

with equation 1 and 2 we obtain $2\lambda y = x(2\lambda x)$

$\therefore 2\lambda y = 2\lambda x^2$. Now $\lambda \neq 0$ yields $y = x^2$ So (3) becomes $x^2 + (x^2)^2 = 2$

$$\text{ie. } (x^2)^2 + x^2 - 2 = 0$$

$$\text{ie. } (x^2 + 2)(x^2 - 1) = 0$$

$$\text{ie. } (x^2 + 2)(x - 1)(x + 1) = 0$$

\uparrow
Negative Discriminant $\therefore x = 1$ or $x = -1$

Note λ does not matter for $f(x, y)$ as long as there is one and is constant.

$$\therefore x = -1: \text{ Then } y = (-1)^2 = 1 \text{ and } \lambda = \frac{e^y}{2x} = \frac{e^1}{2(-1)} = -\frac{e}{2}$$

So $(-1, 1)$ is a potential extreme point of f subject to $g = 0$

$$f(-1, 1) = (-1)e^1 = -e$$

$x = 1: y = 1^2 = 1, \lambda = \frac{e^1}{2 \cdot 1} = \frac{e}{2} \therefore (1, 1)$ is a potential extreme point of f subject to g , and $f(1, 1) = 1e^1 = e$.

$\therefore -e$ is the global min and e is the global max of f subject to $x^2 + y^2 = 2$.